

Supplement to
NUMERICAL SCHEMES FOR CONSERVATION
LAWS VIA HAMILTON-JACOBI EQUATIONS

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Table 1. Test 1: maximum L^1 errors and CPU times.

- (a) (H)= Harten scheme, (G)= Godunov scheme, (HJ)= H-J scheme with $k \leq h$
 (b) H-J scheme $h = k^{3/2}$

Time Step	Mesh Size	L^1 Error	CPU
$k = 0.066666$	$h = 0.300000$	(H) 0.446665	(H) 0.75
		(G) 0.474162	(G) 0.68
		(HJ) 0.526606	(HJ) 0.63
$k = 0.033333$	$h = 0.150000$	(H) 0.226174	(H) 1.07
		(G) 0.266013	(G) 0.69
		(HJ) 0.313992	(HJ) 0.58
$k = 0.016666$	$h = 0.075000$	(H) 0.116579	(H) 2.38
		(G) 0.173682	(G) 0.83
		(HJ) 0.177846	(HJ) 0.78
$k = 0.008333$	$h = 0.037500$	(H) 0.056856	(H) 6.40
		(G) 0.091962	(G) 1.22
		(HJ) 0.103086	(HJ) 1.13

(a)

Time Step	Mesh Size	L^1 Error	CPU
$k = 0.200$	$h = 0.089442$	0.154003	00.75
$k = 0.100$	$h = 0.031622$	0.055008	01.33
$k = 0.080$	$h = 0.022627$	0.040224	01.55
$k = 0.050$	$h = 0.011180$	0.021983	04.56
$k = 0.040$	$h = 0.008000$	0.014811	05.63
$k = 0.025$	$h = 0.003952$	0.008301	29.91
$k = 0.020$	$h = 0.002828$	0.006038	57.19

(b)

Table 2. Test 1: order of convergence in L^1 .

(a) (H)= Harten scheme, (G)= Godunov scheme, (HJ)= H-J scheme with $k \leq h$
 (b) H-J scheme $h = k^{2/3}$

Time Step	Mesh Size	L^1 Order
$k = 0.066666$	$h = 0.300000$	(H) 0.981762
	$h = 0.150000$	(G) 0.833885
	$h = 0.075000$	(HJ) 0.745996
$k = 0.016666$	$h = 0.075000$	(H) 1.035917
	$h = 0.037500$	(G) 0.917328
	$h = 0.018750$	(HJ) 0.786777

(a)

Time Step	Mesh Size	L^1 Order
$k = 0.400000$	$h = 0.252982$	
	$h = 0.031622$	0.944003
	$h = 0.100000$	
$k = 0.100000$	$h = 0.031622$	
	$h = 0.003952$	0.908383
	$h = 0.025000$	
$k = 0.200000$	$h = 0.089442$	
	$h = 0.011180$	0.936154
	$h = 0.050000$	
$k = 0.133333$	$h = 0.046886$	
	$h = 0.006085$	0.927146
	$h = 0.033333$	
$k = 0.079999$	$h = 0.022627$	
	$h = 0.002828$	0.911956
	$h = 0.020000$	

(b)

Table 3. Test 2: maximum L^1 and L^∞ errors, CPU times.

(a) (H)= Harten scheme, (G)= Godunov scheme, (HJ)= H-J scheme with $k \leq h$
 (b) H-J scheme $h = k^{2/3}$

Time Step	Mesh Size	L^1 Error	L^∞ Error	CPU	
$k = 0.015555$	$h = 0.066666$	(H) 0.074652 (G) 0.096322 (HJ) 0.109842	(H) 0.333935 (G) 0.288599 (HJ) 0.379194	(H) 01.05 (G) 00.64 (HJ) 00.63	
	$k = 0.007777$	$h = 0.033333$	(H) 0.120863 (G) 0.145124 (HJ) 0.063475	(H) 0.809503 (G) 0.883329 (HJ) 0.435257	(H) 02.20 (G) 00.86 (HJ) 00.72
		$k = 0.005185$	$h = 0.022222$	(H) 0.176984 (G) 0.174042 (HJ) 0.050141	(H) 0.750002 (G) 0.723084 (HJ) 0.364255
$k = 0.002592$			$h = 0.011111$	(H) 0.094184 (G) 0.097101 (HJ) 0.028279	(H) 0.809735 (G) 0.865321 (HJ) 0.439552
	$k = 0.001728$		$h = 0.007407$	(H) 0.061123 (G) 0.065008 (HJ) 0.021666	(H) 0.750002 (G) 0.723084 (HJ) 0.359344
		$k = 0.000864$	$h = 0.003703$	(H) 0.031589 (G) 0.034907 (HJ) 0.012128	(H) 0.809765 (G) 0.868275 (HJ) 0.440908
$k = 0.000777$			$h = 0.003333$	(H) 0.028432 (G) 0.0311631 (HJ) 0.011184	(H) 0.809823 (G) 0.883329 (HJ) 0.441031

(a)

Time Step	Mesh Size	L^1 Error	L^∞ Error	CPU
$k = 0.140000$	$h = 0.052383$	0.035730	0.217985	00.55
	$h = 0.018520$	0.013746	0.113872	00.78
$k = 0.070000$	$h = 0.006547$	0.005832	0.076641	02.01
	$h = 0.004685$	0.004809	0.095924	03.25
$k = 0.027999$	$h = 0.002315$	0.002662	0.083129	10.96
	$h = 0.001656$	0.002014	0.068289	22.00

(b)

Table 4. Test 2: orders of convergence in L^1 and L^∞ .

(a) (H)= Harten scheme, (G)= Godunov scheme, (HJ)= H-J scheme with $k \leq h$
 (b) H-J scheme $h = k^{3/2}$

Time Step	Mesh Size	L^1 Order	L^∞ Order
$k = 0.015555$	$h = 0.066666$	(H) -0.695108	(H) -1.277470
	$h = 0.033333$	(G) -0.891356	(G) -1.613883
	$h = 0.016667$	(HJ) 0.791155	(HJ) -0.199083
$k = 0.005185$	$h = 0.022222$	(H) 0.910066	(H) -0.110556
	$h = 0.011111$	(G) 0.841877	(G) -0.288775
	$h = 0.005592$	(HJ) 0.826283	(HJ) -0.270994
$k = 0.001728$	$h = 0.007407$	(H) 0.952295	(H) -0.110609
	$h = 0.003703$	(G) 0.897092	(G) -0.286699
	$h = 0.001864$	(HJ) 0.837056	(HJ) -0.295109

(a)

Time Step	Mesh Size	L^1 Order	L^∞ Order
$k = 0.140000$	$h = 0.052383$		
	$h = 0.006547$	0.871626	0.502678
$k = 0.070000$	$h = 0.018520$		
	$h = 0.002315$	0.789439	0.151329

(b)

Table 5. Test 3: L^1 and L^∞ errors, CPU times.

(a) (H)= Harten scheme, (G)= Godunov scheme, (HJ)= H-J scheme with $k \leq h$
 (b) H-J scheme $h = k^{3/2}$

Time Step	Mesh Size	L^1 Error	L^∞ Error	CPU
$k = 0.019999$	$h = 0.125000$	(H) 0.097782	(H) 0.118038	(H) 02.39
	$h = 0.062500$	(G) 0.124811	(G) 0.208365	(G) 02.32
	$h = 0.031250$	(HJ) 0.145058	(HJ) 0.179904	(HJ) 02.17
$k = 0.009999$	$h = 0.062500$	(H) 0.027789	(H) 0.039532	(H) 02.55
	$h = 0.031250$	(G) 0.063864	(G) 0.160217	(G) 02.61
	$h = 0.015625$	(HJ) 0.069323	(HJ) 0.136892	(HJ) 02.35
$k = 0.006666$	$h = 0.100000$	(H) 0.132006	(H) 0.150769	(H) 02.61
	$h = 0.050000$	(G) 0.108680	(G) 0.142837	(G) 02.54
	$h = 0.025000$	(HJ) 0.121180	(HJ) 0.178164	(HJ) 02.26
$k = 0.003333$	$h = 0.050000$	(H) 0.037032	(H) 0.056079	(H) 02.84
	$h = 0.025000$	(G) 0.058254	(G) 0.154768	(G) 02.66
	$h = 0.012500$	(HJ) 0.060456	(HJ) 0.121693	(HJ) 02.21
$k = 0.004999$	$h = 0.031250$	(H) 0.007301	(H) 0.016047	(H) 03.01
	$h = 0.015625$	(G) 0.035587	(G) 0.100405	(G) 02.26
	$h = 0.007812$	(HJ) 0.035350	(HJ) 0.100006	(HJ) 02.31
$k = 0.002222$	$h = 0.033333$	(H) 0.017050	(H) 0.033423	(H) 03.38
	$h = 0.016667$	(G) 0.040112	(G) 0.117632	(G) 02.52
	$h = 0.008333$	(HJ) 0.041297	(HJ) 0.117505	(HJ) 02.20

(a)

Time Step	Mesh Size	L^1 Error	L^∞ Error	CPU
$k = 0.100000$	$h = 0.031622$	0.010208	0.015478	02.28
	$h = 0.011180$	0.003303	0.005853	02.48
$k = 0.025000$	$h = 0.003952$	0.001535	0.006298	03.29
	$h = 0.001397$	0.000646	0.003585	10.87
$k = 0.010000$	$h = 0.001000$	0.000496	0.002740	19.27

(b)

Table 7. Test 4. L^1 and L^∞ errors, CPU times.

(a) (H)= Harten scheme, (G)= Godunov scheme, (HJ)= H-J scheme with $k \leq h$
 (b) H-J scheme $h = k^{3/2}$

Time Step	Mesh Size	L^1 Error	L^∞ Global	L^∞ Smooth	CPU
$k = 0.025000$	$h = 0.200000$	(H) 0.301617	(H) 0.494689	(H) 0.494689	(H) 02.66
		(G) 0.211190	(G) 0.442561	(G) 0.097371	(G) 02.51
		(HJ) 0.255444	(HJ) 0.561811	(HJ) 0.090851	(HJ) 02.33
$k = 0.012500$	$h = 0.100000$	(H) 0.163953	(H) 0.726112	(H) 0.235650	(H) 02.78
		(G) 0.130648	(G) 0.493891	(G) 0.050158	(G) 02.57
		(HJ) 0.140167	(HJ) 0.545271	(HJ) 0.048922	(HJ) 02.32
$k = 0.033333$	$h = 0.200000$	(H) 0.277596	(H) 0.448493	(H) 0.448493	(H) 02.45
		(G) 0.265621	(G) 0.420861	(G) 0.095783	(G) 02.42
		(HJ) 0.251130	(HJ) 0.594391	(HJ) 0.088778	(HJ) 02.19
$k = 0.016666$	$h = 0.100000$	(H) 0.136836	(H) 0.724408	(H) 0.182314	(H) 02.57
		(G) 0.128222	(G) 0.503999	(G) 0.049416	(G) 02.43
		(HJ) 0.137778	(HJ) 0.553873	(HJ) 0.047813	(HJ) 02.31
$k = 0.008333$	$h = 0.050000$	(H) 0.062032	(H) 0.727735	(H) 0.021229	(H) 03.18
		(G) 0.066098	(G) 0.493809	(G) 0.024701	(G) 02.55
		(HJ) 0.074330	(HJ) 0.551196	(HJ) 0.025106	(HJ) 02.42
$k = 0.009999$	$h = 0.066666$	(H) 0.030523	(H) 0.728981	(H) 0.008959	(H) 05.84
		(G) 0.033675	(G) 0.490500	(G) 0.012541	(G) 02.63
		(HJ) 0.100595	(HJ) 0.571155	(HJ) 0.033279	(HJ) 02.42

(a)

Table 6. Test 3. Orders of convergence in L^1 and L^∞ .

(a) (a) (H)= Harten scheme, (G)= Godunov scheme, (HJ)= H-J scheme with $k \leq h$
 (b) H-J scheme $h = k^{3/2}$

Time Step	Mesh Size	L^1 Order	L^∞ Order
$k = 0.020000$	$h = 0.125000$	(H) 1.815059	(H) 1.578137
		(G) 0.966656	(G) 0.379089
		(HJ) 1.065220	(HJ) 0.402644
$k = 0.006666$	$h = 0.100000$	(H) 1.833726	(H) 1.426791
		(G) 0.896651	(G) -0.11574
		(HJ) 1.003184	(HJ) 0.549961
$k = 0.002222$	$h = 0.033333$	(H) 1.779225	(H) 1.286056
		(G) 0.910252	(G) 0.886473
		(HJ) 0.931571	(HJ) 0.871742

(a)

Time Step	Mesh Size	L^1 Order	L^∞ Order
$k = 0.100000$	$h = 0.031622$		
	$h = 0.003952$	0.910904	0.432407
$k = 0.050000$	$h = 0.011180$		
	$h = 0.001397$	0.784666	0.235732

(b)

Time Step	Mesh Size	L^1 Error	L^∞ Error	L^∞ Smooth	CPU
$k = 0.500000$	$h = 0.353553$	0.334314	0.617023	0.193665	02.34
	$h = 0.125000$	0.131575	0.736483	0.040660	02.36
$k = 0.125000$	$h = 0.044194$	0.049307	0.693513	0.018888	02.51
	$h = 0.015625$	0.018441	0.736487	0.007307	03.03
$k = 0.050000$	$h = 0.011180$	0.013252	0.629915	0.005347	03.49

(b)

Table 9. Test 3: large time-step algorithm.

- (a) L^1 error
- (b) L^∞ error
- (c) CPU times

	$h = 0.100$	$h = 0.050$	$h = 0.009$	$h = 0.001$
$k = 0.100$	0.060177	0.019714	0.001897	0.000213
$k = 0.050$	0.088318	0.027463	0.002534	0.000269
$k = 0.025$	0.110049	0.041447	0.003718	0.000389
$k = 0.012$	0.120497	0.052680	0.004787	0.000483

(a)

	$h = 0.100$	$h = 0.050$	$h = 0.009$	$h = 0.001$
$k = 0.100$	0.101660	0.024451	0.002767	0.000380
$k = 0.050$	0.050000	0.049996	0.005860	0.000612
$k = 0.025$	0.159443	0.082033	0.014227	0.001512
$k = 0.012$	0.172254	0.103820	0.019605	0.002590

(b)

	$h = 0.100$	$h = 0.050$	$h = 0.009$	$h = 0.001$
$k = 0.100$	02.24	02.34	02.48	17.18
$k = 0.050$	02.32	02.33	02.61	18.56
$k = 0.025$	02.35	02.35	02.58	19.81
$k = 0.012$	02.23	02.39	02.66	19.79

(c)

Table 8. Test 4: orders of convergence in L^1 and L^∞ .

- (a) (H) = Harten scheme, (G) = Godunov scheme, (HJ) = H-J scheme with $k \leq h$
- (b) H-J scheme $h = k^{3/2}$

Time Step	Mesh Size	L^1 Global	L^∞ Smooth
$k = 0.025000$	$h = 0.200000$	(H) 0.879432	(H) 1.068650
	$h = 0.100000$	(G) 0.692854	(G) 0.956999
	$h = 0.050000$	(HJ) 0.865854	(HJ) 0.893022
$k = 0.033333$	$h = 0.200000$	(H) 1.019498	(H) 1.298657
	$h = 0.100000$	(G) 0.681342	(G) 0.954803
	$h = 0.050000$	(HJ) 0.866085	(HJ) 0.892784
$k = 0.008333$	$h = 0.050000$	(H) 1.023098	(H) 1.244534
	$h = 0.025000$	(G) 0.972890	(G) 0.983127
	$h = 0.012500$	(HJ) 0.953084	(HJ) 0.999432
$k = 0.010000$	$h = 0.066666$	(H) 0.887080	(H) 2.559758
	$h = 0.033333$	(G) 0.898340	(G) 0.977570
	$h = 0.016666$	(HJ) 0.975854	(HJ) 0.985346

(a)

Time Step	Mesh Size	L^1 Global	L^∞ Smooth
$k = 0.500000$	$h = 0.353553$		
	$h = 0.044194$	0.920444	1.118318
$k = 0.250000$	$h = 0.125000$		
	$h = 0.015625$	0.944964	0.825409

(b)

Table 10. Test 3: Order in L^1 and L^∞ for large time-step.

Time Step	Mesh Size	L^1 Error	L^∞ Error
k=0.100000	$h = 0.100000$	1.609143	2.050467
	$h = 0.050000$		0.903300
	$h = 0.010000$		
	$h = 0.001000$		
k=0.050000	$h = 0.100000$	1.433511	
	$h = 0.050000$	1.685221	
	$h = 0.010000$	0.987523	1.005329
	$h = 0.001000$		
k=0.025000	$h = 0.100000$	1.408796	0.958764
	$h = 0.050000$		
	$h = 0.010000$	1.026617	1.020206
	$h = 0.001000$		
k=0.012500	$h = 0.100000$	1.193668	0.730452
	$h = 0.050000$		
	$h = 0.010000$	1.043673	0.925793
	$h = 0.001000$		

Table 11. Test 4: large time-step algorithm.

(a) L^1 error

(b) L^∞ error

(c) L^∞ error in "smooth" regions

(d) CPU times

	$h = 0.100$	$h = 0.050$	$h = 0.009$	$h = 0.001$
$k = 0.50$	0.091143	0.046292	0.008278	0.000923
$k = 0.25$	0.104744	0.051385	0.009246	0.001045
$k = 0.125$	0.113463	0.058492	0.010165	0.001151
$k = 0.062$	0.126772	0.059950	0.010597	0.001207

(a)

	$h = 0.100$	$h = 0.050$	$h = 0.009$	$h = 0.001$
$k = 0.50$	0.736483	0.736482	0.491179	0.491623
$k = 0.25$	0.736417	0.735441	0.736483	0.491928
$k = 0.125$	0.736587	0.714463	0.765638	0.491928
$k = 0.062$	0.736417	0.663833	0.721230	0.490868

(b)

	$h = 0.100$	$h = 0.050$	$h = 0.009$	$h = 0.001$
$k = 0.50$	0.020309	0.010046	0.001842	0.000243
$k = 0.25$	0.032663	0.012076	0.002952	0.000386
$k = 0.125$	0.041846	0.020526	0.003831	0.000587
$k = 0.062$	0.040116	0.022463	0.004187	0.000722

(c)

	$h = 0.100$	$h = 0.050$	$h = 0.009$	$h = 0.001$
$k = 0.50$	02.27	02.21	03.52	01.44.23
$k = 0.25$	02.32	03.82	04.16	02.04.16
$k = 0.125$	02.27	02.33	03.79	02.01.64
$k = 0.062$	02.12	02.14	03.65	01.56.69

(d)

Table 12. Test 4: Order in L^1 and L^∞ for large time-step.

Time Step	Mesh Size	L^1 Error	L^∞ Smooth
k=0.050000	$h = 0.100000$	0.977372	1.015510
	$h = 0.050000$		
	$h = 0.010000$	0.998164	0.921106
	$h = 0.001000$		
k=0.025000	$h = 0.100000$	1.027447	1.022729
	$h = 0.050000$		
	$h = 0.010000$	0.991838	0.925554
	$h = 0.001000$		
k=0.012500	$h = 0.100000$	0.955906	1.027628
	$h = 0.050000$		
	$h = 0.010000$	1.080395	0.836340
	$h = 0.001000$		